Also solved by H. N. Carleton, J. E. Rowe, L. C. Mathewson, A. H. Holmes, Elijah Swift, O. S. Adams, J. Rosenbaum, Louis Clark, E. E. Whiteford, and H. S. Uhler.

241. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $a^2 + b^2 = c^2$, where a, b, and c are integers, then prove that abc will be a multiple of 60.

Solution by Albert G. Caris, Defiance College.

From the well-known theorem that any integral solution of $a^2 + b^2 = c^2$ may be put in the form $2xy$, $x^2 - y^2$, $x^2 + y^2$, where x and y are integers, it follows immediately that

$$abc = 2xy(x - y)(x + y)(x^2 + y^2).$$

Showing that this product is always a multiple of 3, 4, and 5 is sufficient to prove the proposed problem.

I. We may write $x = 2m - 1$, or $2m$ and $y = 2n - 1$, or $2n$, where m and n are integers. Whenever $x = 2m$, or $y = 2n$, abc is a multiple of 4. In all other cases $x = 2m - 1$ at the same time that $y = 2n - 1$ and consequently, $x - y$, $x + y$, and $x^2 + y^2$ are all multiples of 2. Therefore abc is always a multiple of 4.

II. We may write $x = 3r - 1$, $3r$, or $3r + 1$ and $y = 3s - 1$, $3s$, or $3s + 1$, where r and s are integers. Whenever $x = 3r$, or $y = 3s$, abc is a multiple of 3. The combinations resulting from all other cases may be arranged in the two groups below:

Group A		*Group B*	
x	y	x	y
$3r - 1$	$3s - 1$	$3r - 1$	$3s + 1$
$3r + 1$	$3s + 1$	$3r + 1$	$3s - 1$

From combinations of group A the factor $x - y = 3(r - s)$. From combinations of group B the factor $x + y = 3(r + s)$. Therefore abc is always a multiple of 3.

III. We may write

$$x = 5u - 2,\quad 5u - 1,\quad 5u,\quad 5u + 1,\quad \text{or}\quad 5u + 2$$

and

$$y = 5v - 2,\quad 5v - 1,\quad 5v,\quad 5v + 1,\quad \text{or}\quad 5v + 2,$$

where u and v are integers. Whenever $x = 5u$, or $y = 5v$, abc is a multiple of 5. The combinations resulting from all other cases may be arranged in the three groups below:

Group C		*Group D*		*Group E*	
x	y	x	y	x	y
$5u - 2$	$5v - 2$	$5u - 2$	$5v + 2$	$5u \pm 2$	$5v \pm 1$
$5u - 1$	$5v - 1$	$5u - 1$	$5v + 1$		
$5u + 1$	$5v + 1$	$5u + 1$	$5v - 1$	$5u \pm 1$	$5v \pm 2$
$5u + 2$	$5v + 2$	$5u + 2$	$5v + 2$		

All combinations of group C make $x - y = 5(u - v)$. All combinations of group D make $x + y = 5(u + v)$. All combinations of group E make $x^2 + y^2 = 5(5u^2 \pm 4u + 5v^2 \pm 2v + 1)$ or $5(5u^2 \pm 2u + 5v^2 \pm 4v + 1)$. Therefore, abc is always a multiple of 5. Hence, abc is always divisible by 60.

Also solved by S. A. Corey, A. H. Holmes, Horace Olson, J. W. Clawson, J. Rosenbaum, J. E. Rowe, H. C. Feemster, H. N. Carleton, W. J. Thome, Elijah Swift, and E. E. Whiteford.

243. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine rational values of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.

SOLUTION BY E. B. ESCOTT, Kansas City, Mo.

Let $x^3 + px^2 + qx + r = (x + a)^3$. Expanding and collecting terms

$$(3a - p)x^2 + (3a^2 - q)x + (a^3 - r) = 0. \tag{1}$$

If x is rational, the discriminant must be a square, that is,

$$(3a^2 - q)^2 - 4(3a - p)(a^3 - r) = d^2. \tag{2}$$

One simple solution is $a = p/3$.

Then from (1), we get

$$x = -\frac{a^3 - r}{3a^2 - q} = -\frac{\dfrac{p^3}{27} - r}{\dfrac{p^2}{3} - q} = -\frac{p^3 - 27r}{9(p^2 - 3q)}. \tag{3}$$

Other solutions of (2) may be found by Euler's method. Expanding (2), we have

$$-3a^4 + 4pa^3 - 6qa^2 + 12ra + (q^2 - 4pr) = d^2.$$

If we know one solution, we can usually find as many as desired.

Applying the above results to $x^3 - 8x^2 + 12x - 6$, we have $p = -8$, $q = 12$, $r = -6$. By (3), we have $x = 25/18$. (2) becomes $(3a^2 - 12)^2 - 4(3a + 8)(a^3 + 6) = d^2$; or, expanded,

$$-3a^4 - 32a^3 - 72a^2 - 72a - 48 = d^2.$$

$a = -2$ is a solution, which gives $x = \pm 1$. Let $a = b - 2$. Then,

$$-3b^4 - 8b^3 + 48b^2 - 72b + 16 = d^2 = (kb^2 + lb + m)^2.$$

Expanding,

$$(k^2 + 3)b^4 + 2(kl + 4)b^3 + (2km + l^2 - 48)b^2 + 2(lm + 36)b + (m^2 - 16) = 0.$$

Let $m = 4$, $l = -9$, $k = -33/8$. Then

$$b = -\frac{2(kl + 4)}{k^2 + 3} = -\frac{752}{183} \quad \text{and} \quad a = -\frac{1118}{183}.$$

Substituting in (1), we have

$$x = \frac{51287}{8235} \quad \text{and} \quad \frac{1895}{549}.$$

Another value of a is -8, whence $x = 23/4$ and $11/2$. Also

$$a = -\frac{632}{361}, \quad -\frac{74}{13}, \quad -\frac{5738}{1381}, \quad \cdots$$

The corresponding values of x are easily found.

Also solved by ELIJAH SWIFT, NORMAN ANNING, J. A. COLSON, and J. E. ROWE.

Editorial Note.—The problem in effect is to find rational points on the cubic

$$x^3 + px^2 + qx + r - y^3 = 0.$$

If the discriminant of $x^3 + px^2 + qx + r$ vanishes this cubic is rational and an infinity of rational points may be found by the method of section by a line through the double point. If not it is a cubic of one branch and genus 1. The theory of the rational points has been discussed by H. Poincaré, *Liouville Journal*, 1901, 161.